Physics 525, Condensed Matter

Homework 1

Due Tuesday, 26th September 2006

Jacob Lewis Bourjaily

Problem 1

We are asked to study the penetration of normally incident, linearly polarized—with polarization parallel to the surface—electromagnetic radiation into a conductor using the Drude model. Let the surface be located at $z = 0$, with $z > 0$ vacuum and $z \leq 0$ be a conductor. We may assume that the relaxation rate is large relative to the plasma frequency, so $\omega_p \tau \gg 1$, and that the plasma frequency is large relative to the incident radiation, $\omega\omega_p \ll 1$; we should allow $\omega\tau$ to be arbitrary.

a) Let us first consider the limit of a free electron plasma, where $1/\tau \to 0$. We are to solve for the full pattern of the electric field both within and without the conductor, calculate the skin depth, and determine quantitative skin depth in this approximation for visible light at 6×10^{14}Hz incident on copper.

We begin by reminding ourselves of some simple electrodynamics learned by rote long ago when we took Jackson: for light incident on a surface at $z = 0$, with outward normal $\vec{n}$, the conditions to be imposed at the boundary are that the normal components of $\mathbf{B}$ and $\mathbf{D}$ and the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous. If we express the fields in question as

$$\mathbf{E}_{in} = \mathfrak{Re}\left\{\hat{x}E_{in}e^{-i(k\hat{z}+\omega t)}\right\} \quad \mathbf{E}_{ni} = \mathfrak{Re}\left\{\hat{x}E_{ni}e^{i(k\hat{z}-\omega t)}\right\} \quad \mathbf{E}_{r} = \mathfrak{Re}\left\{\hat{x}E_{r}e^{-i(k'\hat{z}+\omega t)}\right\}, \tag{a.1}$$

where $\mathbf{E}_{ni}$ is the reflected wave and the others are self-evident, then making use of Maxwell's equations to relate $\mathbf{B}_i$ to $\mathbf{E}_i$ then we find these boundary conditions—the ones for $\mathbf{E}$ and $\mathbf{H}$—imply that

$$(\mathbf{E}_{in} + \mathbf{E}_{ni} - \mathbf{E}_r)\wedge\vec{n} = 0 \quad \Longrightarrow \quad E_r = E_{in} + E_{ni}; \tag{a.2}$$

$$\left(\vec{k}\wedge\mathbf{E}_{in} - \vec{k}\wedge\mathbf{E}_{ni} - \vec{k'}\wedge\mathbf{E}_r\right)\wedge\vec{n} = 0 \quad \Longrightarrow \quad k'E_r = k\left(E_{in} - E_{ni}\right). \tag{a.3}$$

Now, Maxwell's equations give us $-\nabla^2\mathbf{E} = \frac{\omega^2}{c^2}\epsilon(\omega)\mathbf{E}$, so $k = \frac{\omega}{c}$ in the vacuum and $k' = \frac{\omega}{c}\sqrt{\epsilon}$ in the medium. This allows us to solve the boundary conditions above rather straight-forwardly in no more than a couple of lines of algebra:

$$\boxed{E_{ni} = E_{in}\frac{1-\sqrt{\epsilon}}{1+\sqrt{\epsilon}} \qquad \text{and} \qquad E_r = E_{in}\frac{2}{1+\sqrt{\epsilon}}.} \tag{a.4}$$

After the above preliminaries, we are ready to perform the more specific challenges of the problem. We can easily find the limit of the expression for $\epsilon(\omega)$ predicted by the Drude model when $1/\tau \to 0$:

$$\begin{aligned}\epsilon(\omega) &= 1 + \frac{4\pi i n e^2\tau}{m\omega(1-i\omega\tau)},\\ &= 1 + \frac{4\pi i n e^2}{m\omega\left(1/\tau - i\omega\right)},\\ &\underset{q/\tau\to0}{\longrightarrow} 1 - \frac{4\pi n e^2}{m\omega}.\end{aligned}$$

For much of the range of light frequency[1], this is a negative, real-valued dielectric constant, which means that light essentially does not penetrate the surface. To see this, recall that $k' = \sqrt{\epsilon}$, so if ϵ is real and negative k' is pure imaginary, which means that the strength of the refracted wave dies exponentially inside the surface. This exactly follows our intuition about plasmas. The skin depth is given by

$$\boxed{\delta = \frac{c}{\omega}\frac{1}{\sqrt{\frac{4\pi n e^2}{m\omega^2} - 1}}.} \tag{a.5}$$

ὅπερ ἔδει ποιῆσαι

[1]The point at which k becomes real for copper (in this approximation, which is crude) is 2.6×10^{15} Hz.

Before we move on, we are to calculate the skin depth of copper in this approximation, using real numbers—a headache to most theorists. To do this we need to choose a consistent set of units. I will use the units for which

$$e^2 = 1.907 \times 10^{-72}\ \mathrm{m}^2 \qquad m_e = 6.764 \times 10^{-58}\ \mathrm{m}.$$

These work out quite well and one finds that

$$\boxed{\delta_{\mathrm{Cu}} = 19\ \mathrm{nm}.} \tag{a.6}$$

b) We are now asked to generalize our work above to the situation where there is scattering in general. We should simplify our expressions as much as possible by keeping only leading terms in ω/ω_p and $1/(\omega_p\tau)$. We are to determine the resulting electric fields for the situation of part **a** above, calculate the absorption coefficient and plot this as a function of $\omega\tau$.

Just in case the grader is keeping a tally, please notice that our solution for the full electric field pattern in part **a** above did not depend on the assumption that there was no scattering, so the result applied exactly.

Before we begin, we should comment that we have found nothing slight of horrendous in this problem. There is little elegance, and in general, everything becomes messy very fast. Let us just clarify our starting point and our goal: we know that in the Drude model

$$\epsilon(\omega) = 1 + \frac{i\omega_p^2\tau}{\omega(1 - i\omega\tau)}, \tag{b.1}$$

and from our course in electrodynamics so many years ago[2] that the absorption coefficient is given by

$$T = \frac{4\mathfrak{Re}\left\{\sqrt{\epsilon}\right\}}{|1+\sqrt{\epsilon}|^2}. \tag{b.2}$$

Let us now begin. We will make repeated use of the fact that $\omega_p\tau \gg 1$ and $\omega_p/\omega \gg 1$. The first instance of this appears in the third line, if you're paying attention. To simplify life a lot, we will define the parameter ξ so that $\sinh\xi = \omega\tau$.

$$\begin{aligned}
\epsilon &= 1 + \frac{i\omega_p^2\tau}{\omega(1-i\omega\tau)}\frac{(1+i\omega\tau)}{(1+i\omega\tau)},\\
&= 1 - \frac{\omega_p^2\tau^2}{1+\sinh^2\xi}\left\{1 - \frac{i}{\sinh\xi}\right\},\\
&\approx -\frac{\omega_p^2\tau^2}{\cosh^2\xi}\left\{1 - \frac{i}{\sinh\xi}\right\},\\
&= \frac{\omega_p^2\tau^2}{\cosh\xi\sinh\xi}e^{i(\theta+\pi)}.
\end{aligned}$$

In the last line, we used some hyperbolic trigonometric identities normalizing ϵ where we have defined the phase $\theta = \mathrm{Arg}\left\{1 - i/\sinh\xi\right\} = \arctan(1/\omega\tau)$.

Now before we jump through the last hoops, it is useful to notice right now that $\mathfrak{Re}\left\{\sqrt{\epsilon}\right\} \propto \omega_p\tau$, so if we are keeping things to order $1/(\omega_p\tau)$, then we need only look at terms in the denominator of the expression for T that are second order at least. Indeed, this means we can drop the $1 + 2\mathfrak{Re}\left\{\sqrt{\epsilon}\right\}$ bit from the denominator, simplifying life enormously. Okay, so with that big approximation made clear, we see directly that

$$\begin{aligned}
T = \frac{4\mathfrak{Re}\left\{\sqrt{\epsilon}\right\}}{|1+\sqrt{\epsilon}|^2} &\approx \frac{4\mathfrak{Re}\left\{\sqrt{\epsilon}\right\}}{|\epsilon|},\\
&= \frac{4\omega_p\tau}{\sqrt{\cosh\xi\sinh\xi}}\cos\left(\frac{1}{2}(\pi - \arctan(1/\omega\tau))\right)\left(\frac{\omega_p^2\tau^2}{\cosh\xi\sinh\xi}\right)^{-1},
\end{aligned}$$

[2] It is not really necessary to quote the result, because considering that the power is $\frac{\epsilon}{2}|\mathbf{E}|^2$, this expression is fairly obvious from our work in part **a**.

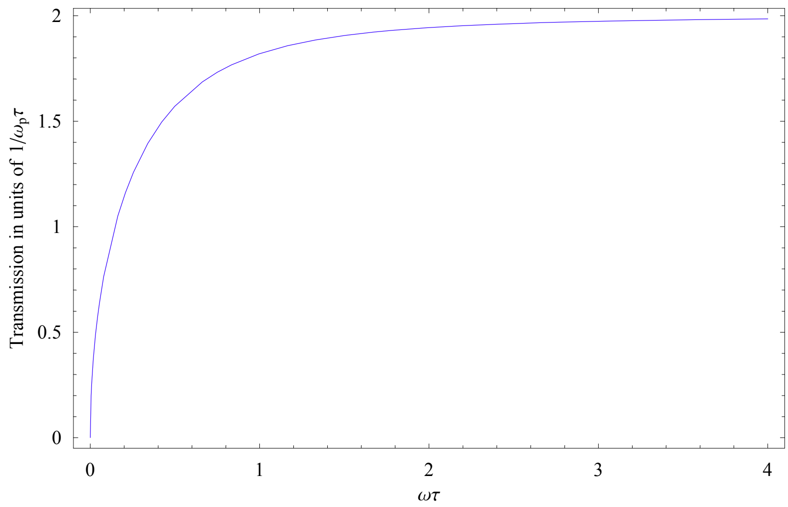

FIGURE 1. The absorption coefficient T as a function of $\omega\tau$ as estimated in problem (1.b).

$$\boxed{\therefore T \simeq \frac{4}{\omega_p\tau}\sqrt{\omega\tau\sqrt{1+\omega^2\tau^2}}\cos\left(\frac{1}{2}(\pi - \arctan(1/\omega\tau))\right).} \tag{b.3}$$

This is shown in Figure 1.

c) We are to compute the skin depth of copper for $\tau = 2.7\times 10^{-14}$ sec using our work above at various frequencies.

From our work above in part **b** it we can easily see that (in the approximation used there)

$$\mathfrak{Im}\left\{\sqrt{\epsilon}\right\} = \frac{\omega_p\tau}{\sqrt{\omega\tau\sqrt{1+\omega^2\tau^2}}}\sin\left(\frac{1}{2}(\pi - \arctan(1/\omega\tau))\right), \tag{c.1}$$

which allows us to write the skin depth

$$\delta = \frac{c}{\omega}\frac{1}{\mathfrak{Im}\left\{\sqrt{\epsilon}\right\}}. \tag{c.2}$$

Using Mathematica so I wouldn't make any silly mistakes, I found the following:

$$\delta_{\mathrm{Cu}}(60\ \mathrm{Hz}) = 8.08\ \mathrm{mm} \qquad \delta_{\mathrm{Cu}}(10^{10}\ \mathrm{Hz}) = 0.625\mu\mathrm{m} \quad \delta_{\mathrm{Cu}}(6\times 10^{14}\ \mathrm{Hz}) = 18.2\ \mathrm{nm}. \tag{c.3}$$

Problem 2

We are to modify the Sommerfeld theory of electrical and thermal conductivity to incorporate two disparate types of scattering events: those with a relaxation time of τ_v which are elastic but thermally randomize the direction of an electron's velocity; and those with a relaxation time of τ_e which fully equilibrate the electron's energy to thermal equilibrium while leaving the speed and direction of the electron unchanged. This is perfectly valid in the limit of temperatures well below the Fermi temperature, because in that case virtually all of the 'effective' conduction electrons are on the Fermi surface and have velocity v_F.

a) We are to compute the electrical conductivity in this two-scattering generalization of the Sommerfeld model.

There are various ways we could make this a bit more rigorous, but our intuition strongly argues that the scatterings which leave the direction of motion unchanged will not contribute to resistance. Indeed, if one were to follow the same type of analysis we did in the one-scattering case, we would find the average velocity at a time dt to be given by

$$\langle\vec{v}(dt)\rangle = \left(1-\frac{dt}{\tau_v}\right)\left(1-\frac{dt}{\tau_e}\right)\left(\langle\vec{v}(t=0)\rangle - e\vec{E}\right) + \frac{dt}{\tau_e}\langle\vec{v}(t=0)\rangle + \mathcal{O}(dt^2), \tag{a.1}$$

where the last term is added because with a probability of $\frac{dt}{\tau_e}$ during the interval dt the electrons can scatter via these inelastic pathways which do not alter the velocity. A quick glance at the equation above shows that this cancels the resistive force caused by the τ_e scattering, so there is no change to our derivation of the electrical conductivity in the original model. Therefore, we see that

$$\boxed{\sigma = \frac{ne^2\tau_v}{m}.} \tag{a.2}$$

b) We are to compare thermal conductivity in this model with the original Sommerfeld model.

Unfortunately, we will need to work a little less rigourously than we would otherwise prefer. Most of the results we can more-or-less guess by considering the symmetries and limits than any solution must have; indeed, it is easy to see that if $1/\tau_e \to 0$ the thermal conductivity will vanish, and similarly if $\tau_v \to 0$; in the first case there are too few inelastic scatterings to transport information about temperature gradients, and in the latter case any thermally interesting transport is washed out by rapid elastic scattering.

Let us first compute the expected scattering time for the combined, independent scattering processes. This is rather straightforward: notice that the probability for an electron to survive until a time t without scattering elastically is e^{-t/τ_v} and the probability to survive until a time t without scattering 'thermally' is e^{-t/τ_e}. Because these are independent random variables, the probability to survive to a time t without any collision is simply the product, or $e^{-t\frac{\tau_e+\tau_v}{\tau_e\tau_v}}$. For this, the differential probability of not scattering is $\frac{\tau_e+\tau_v}{\tau_e\tau_v}e^{-t\frac{\tau_e+\tau_v}{\tau_e\tau_v}}$; and from here the evaluation of an elementary integral shows that the expected time between collisions is

$$\langle t\rangle = \frac{\tau_e\tau_v}{\tau_e+\tau_v}. \tag{a.1}$$

This of course satisfies our intuition because when one of τ_e or τ_v is small, it will always dominate: if on process is much more rapid, the other can be effectively ignored.

Now, a correct derivation would begin by using the fact that only the inelastic scatterings will contribute to the thermal current. One would find something along the lines of

$$
\begin{aligned}
j &= \frac{n}{2}\eta\frac{v_F}{3}\left[\varepsilon(T(x=-v_F\tau_e)) - \varepsilon(T(x=v_F\tau_e))\right], \\
&= \frac{n}{2}\eta\frac{v_F}{3}\left[\varepsilon(T(x=0)) - v_F\tau_e\frac{\partial\varepsilon}{\partial T}\frac{\partial T}{\partial x} - \varepsilon(T(x=0)) - v_F\tau_e\frac{\partial\varepsilon}{\partial T}\frac{\partial T}{\partial x} + \dots\right], \\
&= -\eta\frac{v_F^2}{3}\tau_e\left(n\frac{\partial\varepsilon}{\partial T}\right)\nabla_x T, \\
&= \eta\tau_e\frac{v_F^2}{3}c_v, \\
&= \frac{2\varepsilon_F}{3m}\eta\tau_e\frac{\pi^2}{2}n\frac{k_B^2T}{\varepsilon_F}, \\
&= \frac{\pi^2}{3}\frac{\eta\tau_e k_B^2 T}{m},
\end{aligned}
$$

where we have introduced the parameter η which parameterizes our ignorance (not fundamentally, just the ignorance of the author): η represents the fraction of electrons arriving at x from a given side such that their last scattering was inelastic. A good guess for η would be[3]

$$\eta \stackrel{?}{=} \frac{\tau_v}{\tau_v + \tau_e}. \tag{a.2}$$

At least it has the right properties and limits. If this were the case, then we would find

$$\frac{\kappa}{\sigma T} = \frac{1}{3}\frac{\pi^2 k_B^2}{e^2}\frac{\tau_e}{\tau_e + \tau_v}. \tag{a.3}$$

[3]Note added in revision: this is the right answer.